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In the second place finite groups, in particular multiply transitive groups can regularly be interpreted and frequently with profit, in the geometrical language of a suitably chosen modular space. The abstract group becomes in many cases geometrically intuitive only by reference to an interpretation of this sort. In the third place the subject still presents novelty and has interest on its own account, the methods of investigation are extremely simple, and most of the results for a given modular space are obtained with ease and completeness.

QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

REPLIES.

15 [*1914*, 278; *1916*, 353; *1920*, 114]. In the *Proceedings of the Royal Society of Edinburgh*, Vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

Remarks by the Editor.

A number of communications relative to this question have been received. As regards content they fall into three classes. In the first (and largest) class may be placed mere demonstrations of the algebraic premiss of the question,—that if $x^3 + y^3 = z^3$, then

$$(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3.$$

This is indeed easy, and we are not to suppose that the question implied any difficulty in obtaining an identity within the reach of any high school student who has learned the fundamentals of algebraic notation. It seems necessary to emphasize, therefore, that the truth of the algebraic identity is granted, and that what is desired is an "easy" proof, based on this identity, that the equation $x^3 + y^3 = z^3$ has no solutions in positive integers.

In a second group fall the replies indicating that Tait may have been mistaken. One suggestion is that he may have noted that any of the relations $x = y$, $x = -z$, $y = -z$ will reduce the second equation to an identity, but not the first. This is not impossible; it amounts to supposing that Tait considered the two equations as equivalent,—that he regarded the second as being not only a necessary, but also a sufficient condition for the first. If so, he would not be the only mathematician of note who has confused necessary and sufficient conditions; but such an error would be so obvious in the present instance as to seem rather unlikely. Another correspondent suggests that Tait took for granted that an equation of the form $x^n + y^n = z^n$ cannot hold for an infinite number of sets of values of x, y, z when $n > 1$. Of course, however, this

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of definitions and theorems concerned only with continuity and forming a science which may be called the Analysis Situs of the given space; in the other case there is a corresponding science treating of algebraic relations among figures describable in geometrico-algebraic language.

By employing the term "algebra" in a wide sense to apply to commutative number fields (that is, number fields with commutative multiplication), one may study a "general algebraic geometry," which will include as sub-cases all those distinct instances that arise by further characterizing the commutative number field under consideration. The entire science may be regarded as the study, in geometrical language to be sure, of algebraic relations in a given number of variables defined over a general commutative number field. Theorems in this general algebraic geometry are clearly independent of Analysis Situs since, as has been shown, number fields may be selected in which continuity has no application.

relation *does* hold for an infinite number of cases when $n = 2$; and we can scarcely ascribe to Tait such ignorance of the status of Fermat's theorem as this assumption would imply.

The third class of replies consists of those which seek to show that from a solution of $x^3 + y^3 = z^3$ can be deduced another whose elements are smaller, and hence to apply the method of infinite descent. This is the most reasonable direction in which to look for Tait's "easy" proof. Unfortunately all proofs of this type which have been sent in are fallacious.

30 [*1916*, 88, 354; *1920*, 114]. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

REPLY BY H. E. WEBB, Central High School, Newark, N. J.

An answer to this question is conditioned upon the state of mind of the high-school teachers in question. If they have never taken college mathematics at all, thirty-five hours is as a teacupful of water to a thirsty horse. It should be borne in mind that high-school students often have capacity for reasoning far beyond that with which they are credited; but at the same time that they object to various canonical forms.

A teacher of high-school algebra should have a knowledge of the fundamental principles of number such as are found in the early pages of any good text in the elementary theory of functions of the real variable. He should have also some *clear* notions regarding series, equations generally, and the significance of a complex number system. He will then find that many of the difficulties which his students encounter are due to a dimly realized sense of the need of distinctions which he himself is able to draw clearly. He should also be familiar with the elements of the analytic geometry of the straight line and the conics, in order properly to weigh the value of various topics and examples for the future practical or theoretical development of mathematics. He is presumably familiar with plane trigonometry, as this is now accounted a high school subject.

A teacher of elementary geometry should above all else be familiar with the fundamentals of synthetic projective geometry, together with the logical foundations of metrical Euclidean geometry, in order to escape the embarrassment of finding that his pupils' logic is better than his own.

To bring these various topics into a compass of thirty-five hours calls for a high degree of skill on the part of the instructor. But it may be noted that no great amount of drill in technique is needed, and that many sub-topics which are often presented in college courses may be omitted from an intensive course of this sort.

The college course which seems least of all to bear directly upon secondary mathematics is the calculus, excepting in so far as the derivative may be used in analyzing higher equations. But this seems to many out of place in the high school.

37 [*1919*, 151; *1920*, 115]. Criticize the following as fundamental definitions of elementary geometry:

A *plane surface* is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A *straight line* is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

I. REPLY BY A. A. BENNETT, Baltimore, Md.

The above will be criticized from three aspects,—logical, postulational, and psychological.

From a logical standpoint, the above "definitions" are faulty in that they fail to define. While it is true that a sphere or circle may be viewed abstractly to the extent that the location of the center with respect to other objects need not be specified, caution must be observed when limits are to be used. The limit of a "finite portion" is not definite. As the radius is increased, the finite portion must change, but *how*? If the center remain fixed, a plane or line obtained in accordance with directions could contain no finitely accessible point. If the center move, why might not the subtended angle vary also? By steadily diminishing the angle, the resulting limit might be bounded or even reduce to a point. Unless care be exercised there will be no limit at all. Logical objections (which are numerous) can be met, however, by sufficient care in statement.

Geometry cannot afford to resemble a circle in having neither beginning nor end; nor ought its center of interest be removed "to infinity." Somewhere terms must be introduced for the first time which shall become fundamental for all further developments. While a geometry of inversion may properly start with spheres and circles, Euclidean geometry is usually found to be more tractable when starting with lines and planes. The above "definitions" imply a reversal of practice and leave one wondering as to what shall be done with spheres and circles. Any discussion involving isolated definitions implies a background of a system in which these are to find their place. No extended discussion of the proposed definitions is appropriate until at least a sketch of the development of Euclidean geometry from spheres and circles, the notion of radius, applications of limits, etc., all prior to the mention of lines and planes, is vouchsafed.

Psychologically, curved surfaces and lines appear more complicated than linear ones. Any use of the notion of limits is at least unfortunate in definitions, until the existential character of the object seems to demand it. In the definition of the length of an arc, area of a surface, etc., limits are usually necessary and appropriate. The "incommensurable case" depending upon the existence of irrational numbers, is a different but analogous instance in which limits are sometimes at least appropriate. Even where limits are freely employed few applications would be more objectionable in an elementary geometry on a non-projective basis than the discussion of what happens when a point moves off "to infinity," a region for which the Euclidean geometer has an abhorrence. The circle as the limit of polygons suggests merely an abstraction which can be visualized in a finite portion of space, and even lines were formerly avoided in favor of finite segments.

A careful introduction in a course supplementary to the usual elementary geometry, of the notion of points at infinity, coincident points, tangents as the limit of secants, general and degenerate cases, and incidentally of lines as special members in linear systems of circles—but *not* for the purpose of definition—would surely give the student valuable breadth of vision.

II. Reply by H. E. Webb, Central High School, Newark, N. J.

These definitions are open to many serious objections: In the second definition, the expression "the radius increases" can be interpreted only by employing the straight line concept itself. A beginner is not slow to notice this. The definition also violates the usual assumptions of order for points on a line in Euclidean geometry, and renders impossible many familiar demonstrations. A necessary consequence of the definition would be the equality of radii of two straight lines, since otherwise there would be no laws of congruence in the Euclidean sense. If, however, straight lines are assumed to have equal radii, then any two of them must intersect, and we have an elliptic geometry, which is interesting in its way, but is not Euclidean. From an elementary standpoint it is undesirable to define one configuration as the "limit" of another. Much confusion has in the past resulted from the misuse of the term "limit" in elementary texts. A "limit" should be regarded as a number, and a variable as any one of a class of numbers, all of which are determined according to some law. The idea of a Dedekind cut, explained in simple terms and not too nicely, is easily understood by students who have had two years of elementary mathematics. The numerical measure of the circumference of a circle can be explained in this way. But at the outset of the study of geometry the attempt to define a straight line as the limit of anything else would make a certain unavoidable confusion worse confounded.

Attempts in elementary geometry to define a *straight line* are usually worse than useless.

The first definition could presumably be so reshaped as to be correct, as far as phraseology is concerned, but would be open to the other objections mentioned above; it would also necessitate a restatement of most of the familiar definitions, and would serve no useful purpose as compared with the usually accepted criterion for points on a plane.

III. Remarks by the Editor.

It does not seem certain that the second definition necessitates equal "radii" of straight lines or leads to an elliptic geometry, as indicated in Mr. Webb's reply. In fact, the definition is so vague as to permit a variety of conjectures as to the interpretation which must be made in order to draw conclusions from it. Both the preceding replies exhibit beyond doubt the utterly unsuitable character of the proposed definitions for inclusion in a logical basis of elementary geometry. With respect to the last sentence of Professor Bennett's reply the Editor would express his belief that use of the term "coincident points" should be permitted, if at all, only to the adept; never to the disciple, who may so easily, in this as in other similar cases, mistake an analogy for a proof.

38 [*1919*, 151; *1920*, 115]. As the several courses in secondary and collegiate mathematics are now taught there is a noticeable difference of treatment with respect to the relative emphasis placed on logical accuracy and on development of technique. In elementary algebra, technique predominates, while in plane and solid geometry and advanced algebra logic is more strongly stressed. Trigonometry, analytic geometry, and the calculus are less easily classified; but there is at least a general tendency to emphasize logic in analytic geometry and technique in the calculus. It is suggested that a general appraisal of the reasons for this difference in treatment, its value, and possible alterations, would be helpful.

I. Reply by H. E. Webb, Central High School, Newark, N. J.

This question is very well opened for discussion at this time. It is safe to say that in the teaching of Euclidean plane and solid geometry in secondary schools and colleges there has been an over emphasis upon finer points of logic at certain stages, together with the swallowing of a host of mathematical camels at others. This, more than anything else, has turned students against the subject. Nearly every student has some sense of logical accuracy, and he is, to say the least, annoyed to find it outraged at one moment by what seems a serious break and insulted at the next by a captious insistence upon detail. The difficulty is reducible by careful analysis, and it is earnestly to be hoped that this will be undertaken soon. In the writer's humble opinion the solution, as far as Euclidean geometry is concerned, consists in establishing an arbitrary (or conventional) distinction between projective geometry and metrical geometry, and in the assembling of all of those theorems of projective geometry which are useful for metrical geometry as a list of assumptions with which to begin the latter, in total disregard of any desire to keep this list at a numerical minimum, and without too great fear of redundancy.

In the field of algebra and elementary analysis there is less confusion of the sort referred to, for the simple reason that even the ordinary demands of logic are so often thrown to the winds in favor of a parrot-like repetition of meaningless phrases. The consequent logical absurdities are gracefully "side-stepped" by the instructor. It is a fact that in the development of a subject not all the necessary steps can always be taken. But when such steps are omitted, the omission should be plainly marked. In other words, the instructor may well take a leaf from the book of the student who, upon encountering a particularly difficult demonstration, reads enough of it to obtain a fair idea of its meaning and then passes it by in favor of something which in his judgment is more likely to appear upon the final examination. In doing so he shows an excellent sense of discrimination.

The occasion of the disparity in proportion of time spent upon logical development, on the one hand, and upon drill in technique, on the other, is of course the lack of time for a comprehensive treatment of any of the mathematical branches in the required courses in school and college. This condition is not likely to be corrected immediately. In the field of algebra certainly, and probably also in that of the elementary calculus, emphasis upon technique is the natural result of a mistaken effort to elevate standards of instruction through the medium of written examinations; these should be difficult enough that the student who is bright but lazy cannot "cram" through them. As a result the student who is diligent but less gifted often has a very hard time of it, and the instructor, in a laudable effort to aid and encourage him, loses sight of essential principles, in the organizing of technique. This state of affairs is obviously less easily attained in those branches which lay greater emphasis upon synthesis than upon analysis.

A valuable aid to teachers would be a comparative analysis of relative difficulty of problems in some limited field, based upon the number of distinct principles involved in each problem.

Mathematical principles, which are likely to be remembered, are of much greater worth to teach than tricks of the mathematical trade, which are soon forgotten.

II. Reply by T. G. Rodgers, New Mexico Normal University.

A discussion of this question requires a large experience in both the secondary and collegiate fields of mathematics. Granted this experience, few can speak with equal assurance in both fields. Hence this discussion is confined to the secondary field.

The tendency to stress technique in algebra and logic in geometry seems to grow out of three sources:

1. It is generally believed that technique is more easily acquired by the immature student than logic, and traditionally algebra comes first in the high-school course;
2. Geometric figures form a more concrete basis for logic than algebraic symbols;
3. Elementary geometry has been bequeathed to this modern world as a monument of rigorous logic which too many feel it would be profane to treat in any other way.

From experience, observation and experiment I am convinced that the above tendency, whatever its cause, violates sound principles of teaching. Algebra and geometry should be taught side by side in the high school, not one before the other, and technique and logic should receive about equal stress throughout the course. Technique without logic and logic without action are alike foreign to the adolescent. It must, however, be borne strictly in mind that insistence upon difficult technique causes as many failures as insistence upon rigorous logic; both must be kept within the ability of the high-school student. In a word, the development of technique and logic should receive equal attention; the material for the development of technique should be simplified and enriched by the introduction of geometrical drawing and construction work; the logic sought should be the logic of the high-school student, not that of the expert logician; technique should gradually increase in fineness and logic in rigor as the course advances.

This procedure will aid to develop the power to understand the quantitative relations of the everyday world that confront the student and the ability to put this understanding into successful execution. The procedure of question 38 has a tendency to develop memorists in logic and jugglers in technique.

NEW QUESTIONS.

40. How great emphasis is laid in freshman mathematics upon the elementary algebra of complex numbers? A recent paper by an eminent electrical engineer seems to indicate the need of a knowledge of this subject on the part of draftsmen and mechanics of very limited educational opportunities. The syllabus of the College Entrance Examination Board mentions the topic under the caption "Advanced Algebra," but the question papers call for only the slightest study of numbers of this type. Is Euler's theorem ($e^{i\theta} = \cos\theta + i\sin\theta$) usually presented in college algebra, or is it left to the calculus? Does the topic deserve greater emphasis than it usually gets, for the sake of applications in the field of periodic currents?

41. A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

Every positive integer of the form $8n + 3$ *is the sum of three odd squares.*

Every positive integer is the sum of not more than three triangular numbers.

NOTE. Bachmann[1] states that these theorems have been proved only by the use of the theory of ternary quadratic forms, and refers to the discovery of the theorems by Gauss, and a comparatively simple proof by Dirichlet,[2] by means of this theory.—EDITOR.

DISCUSSIONS.

Professor Noble, discussing the familiar fact that the addition of two rational algebraic fractions each in its lowest terms, by the usual method of reducing to the least common denominator, may lead to a result not in its lowest terms, shows in what way this situation presents itself, and verifies the process suggested by one text-book for securing the result in reduced form. That a similar contingency may arise in arithmetic is clear to anyone who has performed the addition of $\frac{1}{2}$ and $\frac{1}{6}$.

Professor Johnson calls attention to a very simple formula for an approximation to the smaller acute angle of a right triangle in terms of the sides, in which the error is surprisingly small. The formula was given, as the author states,

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[1] *Niedere Zahlentheorie*, Leipsic and Berlin, 1910, Teil 2, p. 325.

[2] *Crelle's Journal*, Vol. 40, p. 228; *Liouville's Journal*, Series 2, Vol. 4, p. 233.